COUNTING HYPERBOLIC MANIFOLDS WITH BOUNDED DIAMETER

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Abstract. Let $\rho_n(V)$ be the number of complete hyperbolic manifolds of dimension n with volume less than V. Burger, Gelander, Lubotzky, and Moses[2] showed that when $n \geq 4$ there exist $a, b > 0$ depending on the dimension such that $aV \log V \leq \log \rho_n(V) \leq bV \log V$, for $V \gg 0$. In this note, we use their methods to bound the number of hyperbolic manifolds with diameter less than d and show that the number grows double-exponentially. Additionally, this bound holds in dimension 3.

1. Introduction

In dimensions 4 and larger, the number of hyperbolic manifolds with a bounded volume is finite. Gelander[3] has proven a similar finiteness result for manifolds locally isometric to symmetric spaces of dimension at least 4, with the exceptions of $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$. For $\mathbb{H}^3$, Thurston[9] has shown that this is not the case and that an infinite number of manifolds of bounded volume can be constructed using Dehn surgery. The diameter of these manifolds, however, becomes large. This note shows that the number of hyperbolic manifolds of bounded diameter is finite and provides upper and lower bounds on its growth. We prove the following bounds.

Theorem 1. *Let $\tau_n(d)$ be the number of closed hyperbolic manifolds of dimension n with diameter less than or equal to d. There exist constants $a, b > 0$ depending on the dimension such that for $d \gg 0$,*

$$e^{e^{ad}} < \tau_n(d) < \begin{cases} e^{bde^{(n-1)d}} & n \geq 4 \\ e^{bde^{5d}} & n = 3 \end{cases}.$$

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2. Constructing the Lower Bound

We achieve the lower bound by constructing exponentially many(in the index of the cover) covers of a non-arithmetic hyperbolic manifold with diameter growing logarithmically in the index. We then claim that only a fraction of these can be isomorphic. This is a modification of the argument given in [2].

We first note that the diameter of a cover can be estimated by its fundamental group as in the following lemma.

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Lemma 2. *Let M be a compact manifold, with $\Gamma = \pi_1(M)$, and $\{\alpha_1, \dots, \alpha_n\}$ a set of generators. Let M' be a finite cover of M with $\pi_1(M') = \Gamma'$ and projection map $p : M' \to M$. There are constants c_1, c_2 depending on M and the choice of generating set such that if $g_1, \dots, g_r \in \Gamma$ are a set such that $\cup_{i=1}^r g_i \Gamma' = \Gamma$ then*

$$\operatorname{diam} M' \le c_1 + c_2 \max\{w(g_i)\}_{i=1}^r,$$

where $w(g_i)$ is the word length of g_i with respect to $\{\alpha_1, \dots, \alpha_n\}$.

Proof. This follows from the observation that Γ is quasi-isometric to the universal cover of M. Indeed, if we fix a basepoint $* \in M'$, and loops $a_i : [0,1] \to M$ based at $*$ and representing the α_i, then one can choose $c_1 = 2 \operatorname{diam} M$ and $c_2 = 2 \max\{\text{length}(a_i)\}$.

Gromov and Piatetski-Shapiro proved in [4] that there exists a non-arithmetic torsion-free cocompact lattice Γ_n in $\operatorname{Isom} \mathbb{H}^n$ for $n \ge 2$. Lubotzky[6] showed that this group contains a finite index subgroup $\Gamma \subset \Gamma_n$ such that there is a surjection $f : \Gamma \to F_2$, where F_2 is the free group on two generators. We will use finite covers of the manifold $M = \mathbb{H}^n/\Gamma$ to establish the lower bound.

Given an index r subgroup G of F_2, we can take the finite sheeted cover $\widehat{M}$ of M induced by the subgroup $f^{-1}(G) \subset \Gamma$. We can then use Lemma 2 to estimate the diameter of $\widehat{M}$. Fix α_1, α_2 generators of F_2; we will construct many finite index subgroups of F_2 with sets of relatively short coset representatives.

We first give a correspondence between certain pairs of permutations of $\{0, \dots, r-1\}$ and subgroups of F_2. Given a pair of permutations (σ_1, σ_2), we can consider the action of F_2 on $\{0, \dots, r-1\}$ where α_i acts by σ_i. If this action is transitive, the subset of F_2 that fixes 0 is a subgroup of index r, and the ith coset is the set of elements taking 0 to i. In addition, if the pair (σ_1', σ_2') determines the same subgroup as (σ_1, σ_2), then there exists a permutation leaving 0 fixed that conjugates σ_1' to σ_1 and σ_2' to σ_2. Conversely, given a finite index subgroup of F_2, the action of F_2 on its cosets by left multiplication gives such a pair of permutations, defined up to relabeling the cosets that are not the original subgroup.

Let σ_1 take i to $i+1 \bmod r$. Let S be the set of permutations σ such that $\sigma(i) = 2i$ for i even, $0 < i < r/2$, and $\sigma_2 \in S$. Then, if $d_k \dots d_1 d_0$ is the binary expansion of $i < r/2$ and $d_k = 1$, then

$$\begin{aligned}
(\sigma_1^{2d_0} \circ \sigma_2 \circ \sigma_1^{2d_1} \circ \dots \circ \sigma_2 \circ \sigma_1^{2d_{k-1}} \circ \sigma_2 \circ \sigma_1^{2d_k})(0)& \\
&= ((\dots((0 + 2d_k) \cdot 2 + 2d_{k-1}) \dots) \cdot 2 + 2d_0) \\
&= 2d_k \cdot 2^k + 2d_{k-1} \cdot 2^{k-1} + \dots + 2d_0 \\
&= 2i
\end{aligned}$$

This gives a coset representative for the $2i$th coset of length at most $3(1 + \log_2 i)$. By composing this with σ_1, we obtain a representative for the $(2i+1)$th coset. Since $|S| = (\lfloor r/2 \rfloor + 1)!$, we can use these to construct $(\lfloor r/2 \rfloor + 1)!$ r-sheeted covers with diameter at most $D \log r$, for some constant $D > 0$. Some of these, however, may be isometric, but we can limit this effect by using the non-arithmeticity of Γ.

Indeed, if two subgroups $\Gamma', \Gamma'' \subset \Gamma$ give rise to isometric covers, then $\Gamma' = \gamma^{-1}\Gamma''\gamma$ for some $\gamma \in \operatorname{Isom} \mathbb{H}^n$ and thus γ is in the commensurator of Γ. Moreover, since Γ is non-arithmetic, by Margulis's Theorem([7] Theorem 1, pg. 2), $[\operatorname{Comm}(\Gamma) : \Gamma] = k < \infty$. Thus a subgroup of Γ of index r is conjugate to at most rk subgroups of Γ, and thus we have constructed at least $\frac{(\lfloor r/2 \rfloor + 1)!}{rk} \ge Ce^r$

non-isometric r-sheeted covers of diameter at most $D\log r$. Equivalently, there are at least $Ce^{e^{d/D}}$ covers of diameter at most d, establishing the lower bound.
Remark: This lower bound can also be established by random methods. Finite index subgroups of a free group F_k correspond to finite covers of a bouquet of k circles, that is, a $2k$-regular graph. If $k \geq 5$, then a random $2k$-regular graph is almost surely an expander and thus has diameter logarithmic in its number of vertices. A proof of this fact can be found in [5].

3. The Upper Bound

For the upper bound, note first that in dimension $n \geq 4$, the argument in [2] holds directly. If M has diameter d, then $\operatorname{vol} M$ is less than the volume of a ball $B_d^{\mathbb{H}^n}$ of radius d in hyperbolic space, and thus, by the theorem in [2], for sufficiently large d,

$$\tau_n(d) \leq \rho_n(B_d^{\mathbb{H}^n}) \leq \rho_n(c_1 e^{(n-1)d}) \leq e^{c_2 d e^{(n-1)d}}$$

where c_1 and c_2 are constants depending on n.

The argument of [2] proceeds by covering M with small balls and using this cover to find $\pi_1(M)$. Since, by Mostow Rigidity, the fundamental group of M determines M up to isometry, a bound on the number of groups obtained this way gives a bound on the number of possible manifolds. In the case of $n \geq 4$, it suffices to find the fundamental group of the thick part of M, however, this fails in dimension 3 because the thin part of the manifold contributes to its fundamental group. In this case, because our manifolds have diameter bounded above and are thus compact, we can find a lower bound on the injectivity radius, which allows us to use similar methods to find a slightly cruder bound.

If M is a closed hyperbolic manifold with injectivity radius ϵ, for ϵ sufficiently small, then its diameter must be at least on the order of $-c\log\epsilon$. M contains a closed geodesic g of length 2ϵ, and for ϵ sufficiently small, g is the central geodesic of a component of the thin part of M. Reznikov[8] shows that the distance between g and the boundary of this component is at least $-c\log\epsilon$, for some constant c, giving a lower bound on the diameter. Conversely, if M has diameter less than d, then its injectivity radius is at least $e^{-d/c}$.

Let M be a hyperbolic 3-manifold of diameter d and injectivity radius r. By the above, $r \geq e^{-d/c}$. If S is a maximal set of points in M such that any two points in S are separated by at least $r/4$, then the set C_M of open balls of radius $r/2$ centered at the points of S covers M and the number of points in S is at most

$$\frac{\operatorname{vol} M}{\operatorname{vol} B_{r/4}^{\mathbb{H}^3}} \leq c_1 \frac{e^{2d}}{(r/4)^3} \leq c_2 e^{5d}.$$

Moreover, each of these balls is convex, so intersections of the balls are convex and thus diffeomorphic to $\mathbb{R}^n$. By Theorem 13.4 of [1], this implies that the fundamental group of a simplicial realization of the nerve of the cover is isomorphic to that of M, so we can calculate $\pi_1(M)$ by considering the combinatorics of this cover.

It suffices to consider the 2-skeleton of the nerve of the cover. The 1-skeleton is a graph whose vertices are the points of S and whose edges are the pairs of points in S that are within r of one another. If $s \in S$ and N_s is the set of neighbors of s, note that since the points of N_s are separated by $r/4$, the $r/8$-balls around points of N_s are disjoint and contained in a ball of radius $r + r/8$ around s. Thus, for r

sufficiently small, the degree of the graph is bounded by a constant k not depending on r; the number of such graphs is at most $|S|^{k|S|} \leq e^{c_3 d e^{5d}}$.

We will bound the number of possible 2-skeleta by considering the number of triangles in such a graph. Since each vertex is adjacent to at most k others, it is a part of at most k^2 triangles, for a total of at most $|S|k^2$ triangles. The number of possible 2-skeleta is then at most $|S|^{(k|S|)} 2^{|S|k^2} \leq e^{c_4 d e^{5d}}$, and thus the number of possible manifolds M is at most $e^{c_4 d e^{5d}}$. $\square$

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